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NEW METHOD FOR SIMULATING FRACTURE USING AN ELASTICALLY UNIFORM RANDOM GEOMETRY LATTICE

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Abstract—This paper discusses 2D lattice models of beams for simulating the fracture of brittle materials. A simulation of an experiment on a concrete plate subjected to shear, in which two overlapping cracks occur, is used to study the effect of individual beam characteristics and different arrangements of the beams in the overall lattice. It was found that any regular orientation of the beams influences the resulting crack patterns. A method is developed to construct a lattice with a random geometry, but which can represent a homogeneous medium, that eliminates the influence of the beam orientation on a simulated crack pattern. Methods to implement a wide range of Poisson's ratios are also developed, and the use of the random lattice to study arbitrary microstructures is outlined. The crack patterns that are obtained with this lattice are in good agreement with the experimental results. Published by Elsevier Science Ltd

1. INTRODUCTION

Heterogeneous materials have complicated fracture mechanisms, which are related to their microstructure. Using linear elastic fracture mechanics to analytically describe these mechanisms is difficult, since the fracture pattern consists of a main crack, with various branches, secondary cracks, and microcracks. Fracture processes in heterogeneous materials are thus often simulated with models. A type of model well suited for fracture simulations is the lattice type. In these models a material is represented as a lattice (or mesh) consisting of an arrangement of small spring or beam elements that can transfer forces. An elastic continuum can be described in this discrete way [1]. The simulation of fracture is achieved by performing a linear elastic analysis of the lattice under loading and removing (or partially removing) an element from the mesh that exceeds a certain threshold, for example tensile stress or elastic energy. The simulation is continued by performing a linear elastic analysis of the new mesh (with one element less), where the forces that were carried by the removed element are now distributed over the surrounding elements. This is followed by the removal of the next element to exceed its threshold, and so on [2].

For heterogeneous materials disorder has to be represented in the lattice in some way. Various methods, depending on the type of material microstructure that is being considered, have been used for this type of modeling. In Refs [2–4] a statistical distribution of material properties was adopted to assign parameters to each of the elements. In Refs [5–8] the material microstructure was directly used to implement the disorder. The materials simulated were concrete or ceramics consisting of aggregates embedded in a matrix. A regular triangular lattice was usually used, which was projected onto an image of the microstructure. Different properties were then assigned to the respective elements that fell inside an aggregate, inside the matrix or on the bond zone. A special application of a lattice model was used in Ref. [9], where a fiber-reinforced material was simulated. Here in the same way as with the aggregates, the properties of the fibers, the matrix and the bond zone were assigned to the respective elements. Another way to implement heterogeneity is by using a random lattice geometry [7, 8, 10–14]. However, in this case the intrinsic disorder of the lattice is not necessarily related to the actual

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heterogeneity of the material one wants to simulate. This will be further discussed in Section 4 of this paper.

The results that are obtained from simulations with lattice models can depend strongly on the chosen element and/or mesh type. In this paper the differences in crack patterns that are obtained from simulations where the element type and mesh are varied are presented. The type of element that gives the most realistic result for a regular lattice is found. However, mesh dependence still exists. Therefore, a method is derived for creating a special lattice with a random geometry that is able to eliminate the mesh dependencies.

In order to give a common base against which to compare the simulation results, all simulations were performed of an experiment in which a plate is loaded in shear (see Fig. 1). The experimental crack pattern plotted in Fig. 1 was obtained from a test on a concrete plate [15]. Concrete is a heterogeneous material. However, the length scale of its inhomogeneity is much smaller than the crack pattern shown in Fig. 1, so that we can assume that the same global overlapping crack pattern will develop in a homogeneous plate under identical loading. Of course, the crack pattern will differ locally, since microcracks caused by the heterogeneity will not be found, and thus also the obtained fracture strain will be somewhat smaller. The global crack pattern will then serve as our test case, against which we will compare the results of various elements arranged in different meshes. Consequently, in the simulations no disorder was implemented, so as to separate the problem of disorder implementation from the ability of a lattice to describe a uniform elastic material.

In Section 2 the elastic equations and solution techniques of the model used in this paper are briefly explained. Section 3 describes differences in simulated crack patterns that are obtained when mesh and element type are varied. The procedure for creating the elastically uniform random lattice is explained in Section 4, and overall conclusions are presented in Section 5.

2. ELASTIC EQUATIONS OF LATTICE MODEL

2.1 Set of equations

In this section the elastic equations of the lattice model with beam elements will be described. Each of the beams in the lattice can transfer, in general, normal force (F), shear force (Q) and bending moment (M) (see Fig. 2b). The relation between these forces and the corresponding displacements (Fig. 2b) for the endpoints of a beam can be expressed as follows:

$$\begin{bmatrix} F_1 \\ Q_1 \\ M_1 \\ F_2 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & \frac{-EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{-6EI}{l^2} & 0 & \frac{-12EI}{l^3} & \frac{-6EI}{l^2} \\ 0 & \frac{-6EI}{l^2} & \frac{4EI}{l} & 0 & \frac{6EI}{l^2} & \frac{2EI}{l} \\ \frac{-EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & \frac{-12EI}{l^3} & \frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{-6EI}{l^2} & \frac{2EI}{l} & 0 & \frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{bmatrix} \quad (1)$$

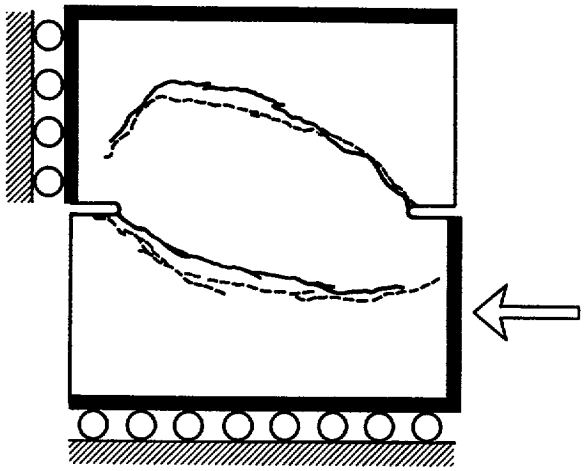
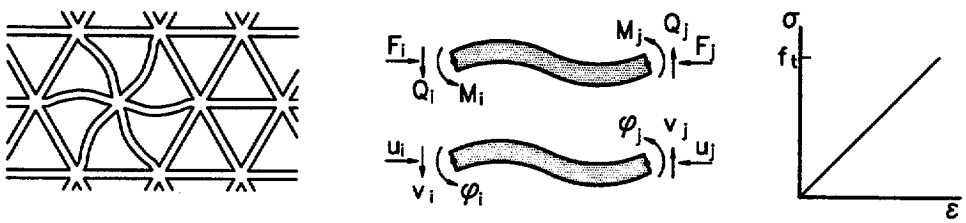


Fig. 1. Geometry and crack pattern of shear experiment on concrete plate, according to Ref. [15].

in which E is the Youngs modulus, l is the length, A is the cross-sectional area, and I is the moment of inertia (a function of the cross-sectional shape) of a beam element, (u, v) are the translational displacements, and ϕ is the rotational displacement. Equation (1) uses simple beam theory [1, 3]. For a lattice with regular geometry, like that shown in Fig. 2a, E, l, A and I are in principle equal for all elements. However these parameters could be varied, element by element or according to a superimposed microstructure, in order to implement heterogeneity.

To construct the system of equations for the complete lattice, the element matrix has to be multiplied with the appropriate rotation matrix, given by equation (2), and positioned correctly in the full system matrix. The rotation matrix is defined as:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$



(a) (b) (c)

Fig. 2. Regular triangular lattice of beams (a), external forces and deformations on a single beam element (b), and stress–strain relation for an element (c).

in which θ is the angle of a beam with the global x-axis. The rotation matrix is 2D, in that it rotates only around the z-axis. The final set of equations for the system is of the form:

$$\mathbf{b} = \mathbf{A}\mathbf{x} \tag{3}$$

in which **b** is the load vector, **A** is the stiffness matrix, and **x** is the displacement vector.

2.2 *Solving the equations*

When solving the set of linear elastic equations for a lattice under an applied load, the load vector and the stiffness matrix are known and the displacement vector is to be determined by solving equation (3). One method to solve the set of equations is to use a direct solver which finds the inverse of **A**, for instance by Gaussian elimination. More information about this procedure can be found in many finite element textbooks [16]. A faster way to find the solution for the set of equations is by the conjugate gradient method [17]. For the simulations in this paper a special version of a conjugate gradient solver [18] was adopted. In this algorithm the displacement vector **x** is solved iteratively by minimizing the functional **G**, which has the dimensions of energy,

$$\mathbf{G} = 0.5\mathbf{x}\mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{x} \tag{4}$$

The advantage of a conjugate gradient solver becomes even larger for fracture simulations. Breaking an element and thus removing it from the lattice is a local effect. This implies that the resulting changes in the deformation vector and in **G** will be small. Therefore only a few iteration steps will be needed to relax the system and find the next element to remove. It should be noted that more iterations of the algorithm are necessary to converge to a solution when the difference in components of the matrix becomes large. This will happen, for example, when simulating a multi-phase composite where one phase is much stiffer than the other phases. In that case preconditioning of the matrix can help to speed up the process [19].

When a direct solver is used the complete system usually has to be re-solved every time an element is removed. However, in this case the method of structural variation could also be used [20, 21]. In this method, the inverse of the matrix **A** is used to update the displacement vector when an element is removed from the mesh, so that a full-scale solution is not needed every time. This does require a large amount of computer memory, however, since the inverse matrix must be stored. Although for a lattice of beams, the stiffness matrix is sparse, the inverse of that matrix is generally not sparse. For a general system of N nodes, the inverse of **A** is of size $(3N)^2$, which can be enormous. Clearly, the method of structural variation will be useful only for small systems.

2.3 *Elastic moduli and fracture criterion*

The 2D elastic moduli tensor for a regular triangular lattice of beams, which is elastically isotropic [22], can easily be derived analytically by evaluating the elastic energy of a unit cell of the lattice under a uniform strain (hydrostatic or simple shear):

bulk modulus

$$\kappa = \frac{\sqrt{3}}{2}\frac{EA}{l}; \tag{5}$$

shear modulus

$$\mu = \frac{\sqrt{3}}{4}\frac{EA}{l}\left(1 + \frac{12I}{Al^2}\right); \tag{6}$$

Poisson's ratio

$$\nu = \frac{\kappa - \mu}{\kappa + \mu} = \frac{\left(1 - \frac{12I}{Al^2}\right)}{\left(3 + \frac{12I}{Al^2}\right)} \quad -1 < \nu < \frac{1}{3}. \tag{7}$$

To simulate fracture a breaking rule must be defined. Different criteria for fracture have been adopted and can be found in the literature [3, 7, 23]. The main idea is that an element of the lattice will break when a predefined threshold for some quantity, for example tensile stress or elastic energy, is exceeded in that element. In the simulations in this paper a beam is removed from the lattice when:

$$\frac{F}{A} > f_t \tag{8}$$

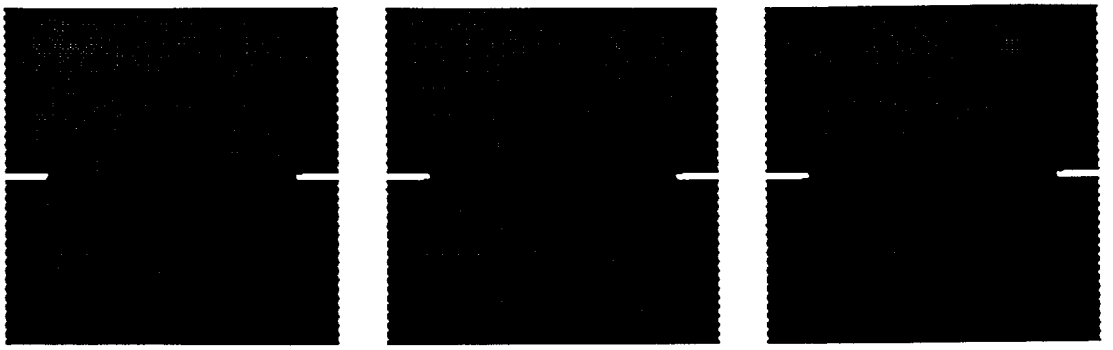
where F is the normal tensile force in the beam, A is the cross-sectional area, and f_t is the tensile strength of the beam. Thus the local behavior of a beam element is perfectly brittle. The stress–strain relation of such an element is plotted in Fig. 2c.

3. ELEMENT AND MESH DEPENDENCY

3.1 Influence of element type on the crack pattern

A lattice of springs or beams is a discretization of a continuum. The number of degrees of freedom in the nodes of a lattice determines the type of continuum that the lattice represents. In this section a comparison is made of results of fracture simulation using regular triangular networks with different kinds of elements. The experiment shown in Fig. 1 is simulated with elements having 1, 2 or 3 degrees of freedom, respectively, in each node. In the first simulation the elements can only transfer normal force, and thus the lattice is equivalent to a central force spring network [4, 8, 12, 13, 24]. The first equation of equation (1), $F = \frac{EA}{l}(u_1 - u_2)$, corresponds to this network. This network is a discretization of a linear elastic continuum with a value of the Poisson's ratio fixed at 1/3. In the second simulation, the elements can also support a shear force, incorporating the first equation and half of the second equation of equation (1), $F = \frac{EA}{l}(u_1 - u_2)$ and $Q = \frac{12EI}{l^3}(v_1 - v_2)$. These elements with normal force and shear force are isomorphic to a spring network with central force plus rotational springs [7, 9, 25]. This second network is a discretization of a linear elastic continuum with a general value of Poisson's ratio (but less than 1/3). In the third case elements with normal force, shear force and bending moments are used [2, 5–7, 26]. Here all three parts of equation (1) must be included. This lattice is a discretization of a higher order continuum, a Cosserat elastic continuum [3, 27], again with a general value of the Poisson's ratio less than 1/3. For a more complete overview of different lattice models the reader is referred to Ref. [3].

The results of the three simulations are shown in Fig. 3. As mentioned before, no heterogeneity was implemented. The crack patterns differ appreciably between the lattices with different elements. In the lattice with only normal force in the elements, one straight crack is obtained. When two degrees of freedom per node are used the crack starts at the correct



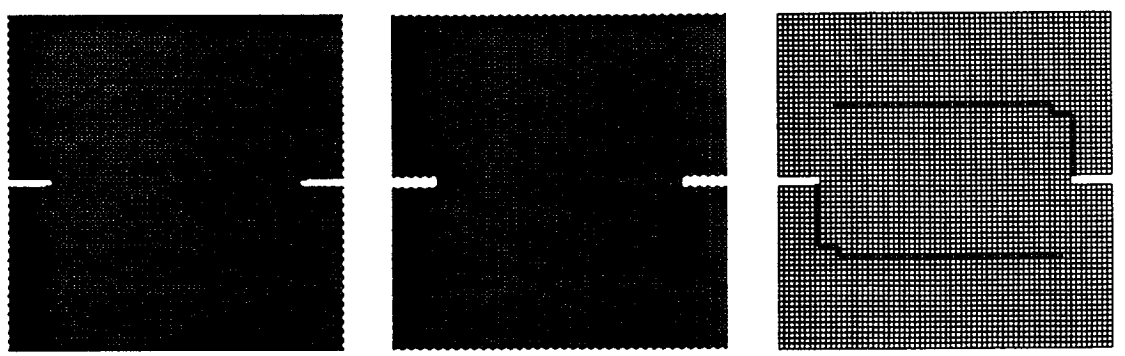
(a) (b) (c)

Fig. 3. Simulated crack patterns in mesh with springs (a), spring and shear elements (b), and beam elements (c).

angle, but after that the crack stays straight and does not curve. In the simulations with the beams with three forces, the crack pattern is much closer to the experimentally obtained cracks. In Ref. [28] it was shown for continuum models that using a Cosserat continuum will in some cases result in more physically realistic crack patterns. However, in all three cases it can also be seen that the crack patterns are influenced by the principal directions of the mesh. We therefore fix on the full beam elastic element as our element of choice, and proceed to illustrate the effect of mesh orientation on simulated crack patterns.

3.2 Influence of mesh orientation on the crack pattern

The effect of mesh orientation on crack patterns when using the full beam elements is illustrated with the next example. In Fig. 4, three crack patterns are shown of simulations of the same experiment using three different lattices. The first lattice, shown in Fig. 4a, is the same as that shown in Fig. 3c. In the second simulation, shown in Fig. 4b, a triangular lattice is also used, but the mesh is rotated 90°. In the third mesh beam elements in a square grid are used (Fig. 4c). In all three simulations two overlapping curved cracks are obtained, yet there are



(a) (b) (c)

Fig. 4. Simulated crack patterns in triangular (a), rotated triangular (b), and square (c) mesh.

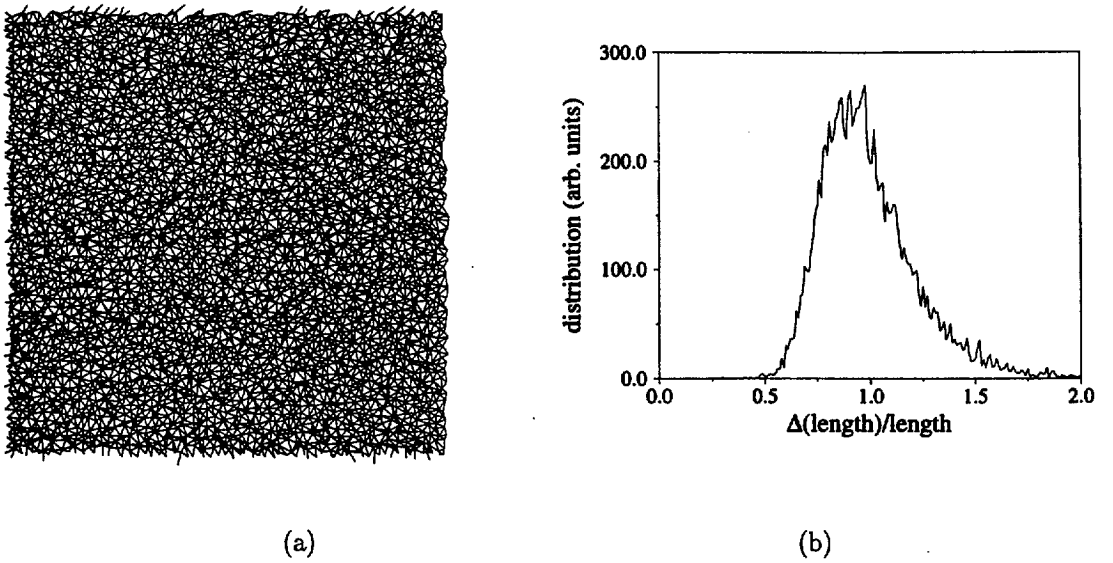


Fig. 5. Random lattice (a), and plot of ratio of Δ (length)/length of lattice under uniform strain (b).

differences in crack shape due to the different orientation of the meshes. In the next section we show how to create a random lattice that can eliminate this mesh dependency.

4. PROCEDURE TO OBTAIN A UNIFORM ELASTIC RANDOM LATTICE

4.1 Explanation of problem

In this section a random lattice that was briefly mentioned in the introduction is adapted to simulate a homogeneous medium (see Fig. 5a). The technique to generate the random lattice that is used in this paper is described in more detail in Ref. [29]. Basically, a collection of nodes are generated according to some statistics. These nodes are then linked together with their near neighbors so that there is an average of six bonds per node. The angles between adjacent bonds and the lengths of the bonds are then random parameters. In previous fracture simulations [10, 11] this lattice was used for implementing disorder by changing each bond into a beam element, all with identical elastic properties, but with the beam length and orientation angle following that of the bond it replaced. The crack patterns that are obtained using this randomly-generated mesh are independent of the particular mesh used. This was not the case when using a regular lattice, as was explained in the previous section. However, a problem does exist with this lattice.

We wish to think of a given lattice, when no disorder has been added, as representing a uniform elastic continuum. A uniform elastic continuum is defined operationally by the fact that a uniform applied strain results in a uniform stress, and vice versa. For a regular lattice, this is achieved by the fact that each node is at a center of symmetry, and all beams are the same, so that for any uniform applied strain, the loads in the beams at any nodes sum to zero. The lattice is in equilibrium under a uniform strain, so that a uniform stress results, and no extra displacements of the nodes are necessary. When a random lattice (randomly positioned nodes) has a uniform strain applied, the nodes are not at a center of symmetry, so nodes have to move to achieve equilibrium, resulting in a non-uniform stress field. The extra displacements result in a non-uniform displacement field as well. This happens for a general assignment of elastic properties for each beam element, and not just for the case where each beam has identical elastic properties.

To get an idea of the effect of the random lattice on the deformations, the test shown in Fig. 5 was performed. A lattice of $(64)^2$ nodes with periodic boundary conditions (see Fig. 5a) was given a uniform strain equal to 1. This means that, if the lattice really did represent a uniform elastic medium, the length of all the beams would simply double. In Fig. 5b the distribution of the actual change in length over the original (Δlength/length) for each beam is plotted. The values vary between 0 and 2 with an average value equal to about 1.

Previously, it has been noted that the distribution of the extensions of the elements looks similar to the distribution of the original lengths of the elements. This observation has led to the speculation that there was a local relation between the length and orientation of the elements and the deformation of the connecting nodes. One could think of changing each beam element's elastic properties in order to make the lattice effectively elastically uniform, using some sort of rule involving the lengths and angles of the local beams. Using a Voronoi construction applied to a random set of points, Jagota and co-workers [7] have tried several scalings between the length of elements and the size of Voronoi cells, as suggested in Ref. [25], to try to predict what the cross-section (A) and the moment of inertia (I) of the beams must be in order to make the lattice elastically uniform. This attempt failed. The reason why all such local rule attempts fail will be discussed in Section 4.5 below.

In the next section a procedure will be described that can make a random geometry lattice elastically homogeneous, so that a uniform applied strain results in a uniform applied stress and a uniform displacement field. There will also then be no dependence of crack patterns on the mesh. This mesh can be used to model general elastic properties. Disorder can then be implemented by using a grain structure like the way it is done for a regular lattice, as was explained in Section 1.

4.2 Technique used

The main idea of this technique is to find a way to systematically choose a set of As and Is that will cause the random lattice to act as if it were elastically uniform. The parameters that can be varied in the random geometry lattice to obtain a lattice that represents a homogeneous medium are the parameters that control the tension, shear and bending components, i.e. the cross-section (A) and the moment of inertia (I). Allowing these to vary is equivalent to allowing the cross-sectional area and shape of each beam to vary. The lengths of the beams are fixed by the random lattice construction, and the Young's modulus of each beam is taken to be the same. In Section 2 the load–displacement equations for the beam model were given. For each node there are three equations, which means that a lattice with N nodes has $3N$ equations. These equations are independent, since there are $3N$ degrees of freedom. A random lattice of the kind we are discussing that has N nodes and periodic boundary conditions is constructed to have $3N$ beam elements. Since each element has an A and an I, there are $6N$ parameters for the lattice that can be varied to obtain a homogeneous lattice. The deformations and loads of a homogeneous medium are known. Each node's displacements should simply be given by the applied strain and the position of the node, i.e. $\mathbf{u}(r) = \bar{\bar{\epsilon}} \cdot \mathbf{r}$, where $\mathbf{r}$ is the position vector of the unstrained node, $\mathbf{u}$ is the displacement vector of this node, and $\bar{\bar{\epsilon}}$ is the (uniform) strain tensor. This means the load vector and displacement vector are known and can be inserted in equation (1). Equation (3), since it is linear in the As and Is, can then be rewritten as:

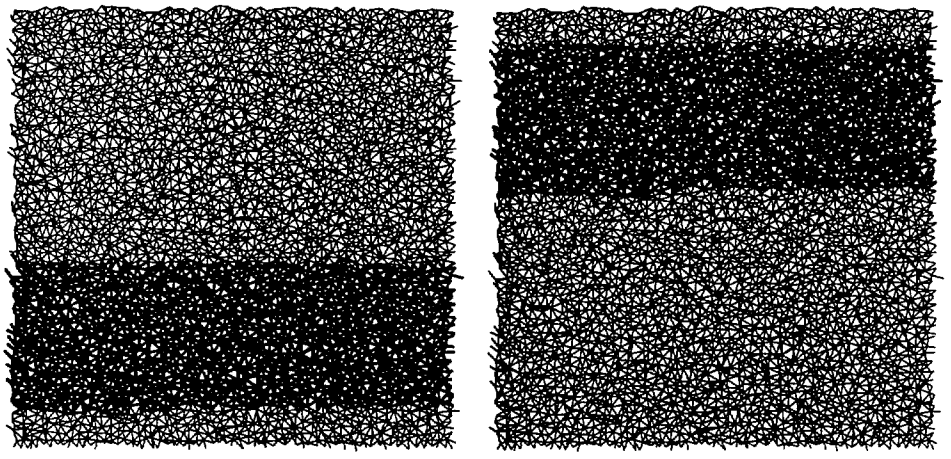
$$\mathbf{b} = \mathbf{X}\mathbf{a} \tag{9}$$

in which $\mathbf{b}$ is again the load vector, $\mathbf{X}$ is a matrix containing the (known) displacements, and $\mathbf{a}$ is the unknown vector containing the values of A and I for all the beams. The load vector contains $3N$ values, the matrix is of size $3N \cdot 6N$ and the unknown vector is of size $6N$. This set of equations has more unknowns than equations, so is underdetermined. There are then many solutions, of which we only need to find one. In principle, if all the values of A and I were independent, one could simply set $3N$ of them to an arbitrary value, treating them as

parameters, and then solve for the rest using the $3N$ equations. However, the process is not that simple.

The main difficulty is that all the values of A and I are not independent. Consider a single node, attached to six other nodes by beam elements. The load and displacement are assumed to be known at that node, and all surrounding nodes. If the parameters of four of the beams are set to a value, then the parameters of the remaining two beams are also fixed, and so are not independent of the others. If we knew how many were independent, say M, and could find a set of these M values, then the others could be given arbitrary values and the M independent ones solved for directly from equation (9). This does not seem to be possible. Therefore, the following trick was used to obtain a valid solution.

First only the cross-sections (A) are solved for in a situation where the moments of inertia (I) of the beams have no influence. If a periodic regular lattice is subjected to a hydrostatic strain all the beams will only elongate, with no shear deformation of the beams or rotation of the nodes. This means that the lattice of beams acts like a spring network. This is what we also require for the random lattice under a hydrostatic strain. For this case only the cross-sections (A) have to be determined. However, a spring network has only 2 degrees of freedom per node (no rotations), which means that a lattice of N nodes has $2N$ equations, with $3N$ elements and thus $3N$ unknown cross-sections. Again the equations can be rewritten as explained above. This results then in $2N$ equations, yet the vector $\mathbf{a}$ is $3N$ long. All $3N$ of the cross-sections are not independent, however, we do not know how to pick the independent ones. There are again many possible solutions for these equations. To find one solution we first choose the values of N of the unknown cross-sections (for instance equal to 1) after which the system can be solved for the remaining $2N$ unknown cross-sections using a conjugate gradient method. It should be noted that the elements that are fixed cannot be chosen randomly, because in that case there will always be some that will be dependent. Too many values would be fixed, so that there would be more equations than independent cross-sections, and so a solution could not be found. In our approach the elements that are fixed are located close to each other, in a uniform band, as shown in Fig. 6a. The elements plotted with thick lines are the elements with cross-sections that are fixed. Of course also in this case too many cross-sections are fixed, because at the edges there will be some extra beams determined due to the N that are fixed. Thus while solving for the remaining $2N$ cross-sections, the iteration process will not converge completely. Because of this and of the fact that the initial N fixed values did not necessarily



(a) (b)

Fig. 6. Random geometry lattice with band of 1/3 of the elements fixed in the first (a), and second step (b).

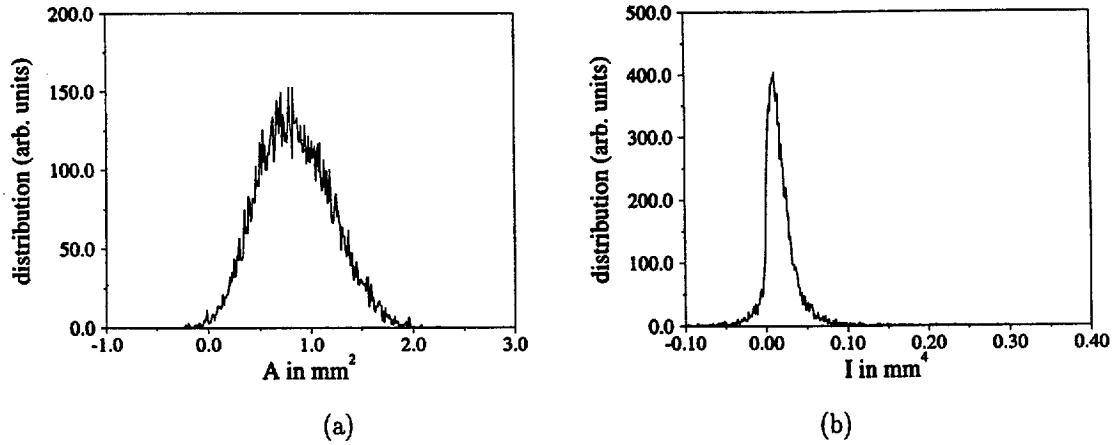


Fig. 7. Distribution of (a) cross-sections (As), and (b) moments of inertia (Is).

satisfy the uniform elastic continuum condition, the resulting values for the other cross-sections are wrong as well. Yet, when the point of no further convergence is reached, the cross-sections of N different elements, also in a band as shown in Fig. 6b, are fixed to the values obtained in the previous step. Again the system of equations is solved for the other cross-sections. The result is still not correct but is already closer to a good answer. Now the first N cross-sections that were fixed are fixed to the values found in the previous step and the system is solved again. After doing this step several times the cross-sections will not change any more, and also the iteration procedure has converged completely, which means a valid solution for the cross-sections is found. This method has avoided the necessity of finding the independent cross-sections.

For the lattice shown in Fig. 6 the distribution of cross-sections that is obtained from this procedure is plotted in Fig. 7a. Note that a small fraction of the cross-sections A turn out to be negative. There may exist solutions in which all the A values are positive, but we have not been able to find any. This small fraction of beams with negative values of A turns out not to matter in actual simulations. We find that the actual tensile stresses, F/A, as in equation (8), are always positive. Further research is needed on whether the solution method can be constrained so as to obtain all positive values of A, or whether such solutions even exist.

Next the values for the moments of inertia (I) for the beams have to be found. For this the lattice can be subjected to a uniaxial strain. Again the load and displacement vectors are known, because the objective is to obtain a lattice which represents a homogeneous medium. The nodal displacements are forced to follow a uniform uniaxial strain. All the cross-sections are known from the previous step. These values can directly be plugged in the equations. The equations also have to be rewritten as explained above. Now we have $3N$ equations with $3N$ unknown moments of inertia. This can be numerically solved directly, which results in the distribution plotted in Fig. 7b for the lattice of Fig. 6. Again note, as in the case of the As, that a small fraction of the I values are negative. It should be noted that even though a particular applied uniaxial strain was used to compute these values of I, the same values make the lattice homogeneous under any applied strain, since any applied strain can be written as a sum of uniaxial and hydrostatic strains, and this lattice was made to be spatially isotropic.

The set of Is that are found depends on the starting values of I that are given to the conjugate gradient solver. This is because the $3N$ equations for the Is are not all independent, so that the solution found is not unique. Another way to vary the resulting Is is by multiplying the set of As with a constant C. This introduces a way to control the Poisson's ratio of the lattice [see equations (10)–(12) below]. If the constant C approaches 0, the Poisson's ratio approaches -1.0. If C goes to plus or minus infinity, the Poisson's ratio goes to 1/3. For intermediate values of C the Poisson's ratio varies between -1.0 and 1/3.

4.3 *Elastic moduli*

The elastic moduli of the homogenized random lattice can then be expressed as follows:

bulk modulus

$$\kappa = \frac{E}{4A_{\mathrm{TOT}}} \sum_{i=1}^{N} A_j l_j \tag{10}$$

shear modulus

$$\mu = \frac{E}{4A_{\mathrm{TOT}}} \sum_{i=1}^{N} \left[\frac{12I_i}{l_i} + \frac{4\alpha_i}{l_i^3} \left(Ai - \frac{12I_i}{l_i^2} \right) \right] \tag{11}$$

Poisson's ratio

$$\nu = \frac{k-\mu}{\kappa+\mu} - 1 < \nu < \frac{1}{3} \tag{12}$$

in which E is the Young's modulus of a beam, A_{TOT} is total area of the lattice and $\alpha_i = (x_1 - x_2)^2 (y_1 - y_2)^2$ for the unstrained endpoints, (x_1, y_1), (x_2, y_2), of the ith beam. The sums are taken over the beam elements. These formulas will work for any lattice homogenized by the procedure presented in this paper. Also, these simple formulas can only be obtained because the lattice is homogeneous, so that an applied strain (affine deformation) results in an affine deformation of the nodes as well. The computed elastic moduli agreed very well with these analytical results.

4.4 *Result for fracture*

With a lattice containing the sets of A and I obtained using the procedure of the previous section, the experiment shown in Fig. 1 was simulated. The resulting crack pattern is plotted in Fig. 8. Two curved overlapping cracks developed in the specimen. The cracks grow in the direction which is perpendicular to the maximum tensile stress in the specimen, are independent of the mesh lines, and agree quite well with the experimental results. The small

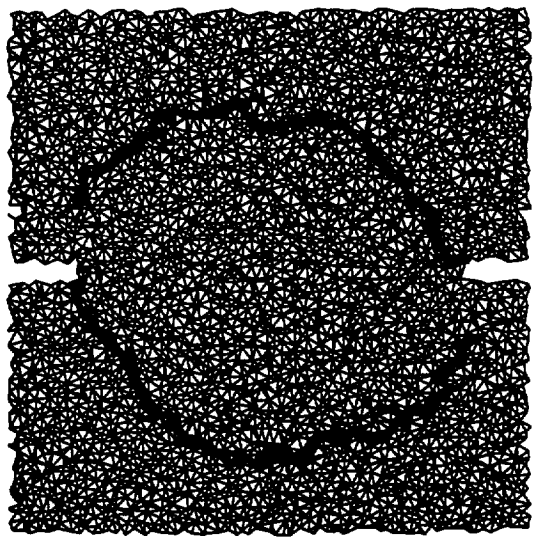
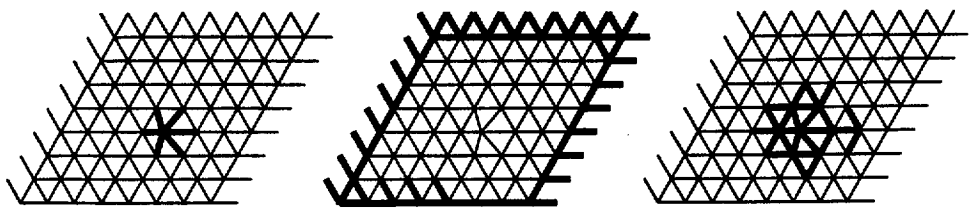


Fig. 8. Crack pattern obtained using the homogenized random lattice.



(a) (b) (c)

Fig. 9. Uniform lattice of beams, with (a) one site slightly shifted, with the affected elements shown in heavy dark lines, (b) heavy dark lines show the elements that were fixed during the first homogenization step, and (c) heavy dark lines indicate those elements whose cross-sections differ by more than 5% from the original value of 1.

fraction of beams with negative values of A and/or I did not result in any uncontrolled local strains or stresses.

4.5 Test with special random lattice

In Section 4.2, a set of cross-sections and moments of inertia was derived to make the lattice of Fig. 6 homogeneous. The distribution of these values were similar to those shown in Fig. 7. It is interesting to test whether a relation can exist between the local geometry of the lattice and the distributions obtained for the values of the As and Is. For this purpose, a special random lattice was created as plotted in Fig. 9. It is a regular triangular lattice with periodic boundary conditions, but with one node shifted slightly. This affects the length and the angle of the six attached elements (see Fig. 9a). If the beam parameters could be predicted by the local geometry, then at most 12 beams would be affected, those beams belonging to the six triangles to which the central shifted node belongs. The same procedure as explained in Section 4.2 was used to make this lattice homogeneous. The elements plotted with thick lines in Fig. 9b were given a value of A equal to 1 in the first homogenization step. In Fig. 9c the elements are plotted with thick lines from which the resulting cross-sections differ more than 5% from 1, yet many other elements had cross-sections different from 1 as well. This means that changing only the length and angle of six elements will result in different cross-sections for many more elements. The same result was also found for the moments of inertia. Changing even one node resulted in changes in cross-sections and moments of inertia far away from the shifted node. From this result it is concluded that there can be no direct relation between local geometry and beam properties. This is the reason that a previous attempt using a Voronoi construction failed, as was discussed above [7].

5. CONCLUSIONS

This paper discussed 2D lattice models for fracture simulations. In the literature models with various types of elements can be found. The equations for the network models with these different elements are all discretizations of different continuum equations. For fracture the results that are obtained depend strongly on the chosen element type. Beam elements with three degrees of freedom per node give the best agreement with experimentally obtained crack patterns. In simulations with different elements, using less degrees of freedom per node, see for instance Refs [8, 12, 21], realistic crack patterns are also obtained. However the crack patterns that are simulated are not as complicated. Uniaxial tensile tests or bending tests are simulated

in which a straight crack surrounded by microcracks develops. In Fig. 3 of this paper it is shown that if the crack pattern is more complex, if the cracks are curved, elements with three degrees of freedom are necessary.

The shape or orientation of the beams in a regular lattice also influences the simulated crack patterns, with the cracks tending to follow the mesh lines. This disadvantage can be circumvented by using a random geometry lattice. However such a lattice also implements an intrinsic disorder, which is unrelated to the disorder of the actual material one is trying to simulate. A method was then presented for making a random geometry lattice elastically homogeneous, to get rid of, to some extent, this intrinsic disorder, yet still retain the mesh independence for crack propagation. A procedure has been developed from which an exact solution for the cross-sections and moments of inertia can be derived. It was found that there is no direct relation between the obtained values for A and I and the local geometry of the lattice. Cracks in such a lattice develop perpendicular to the maximum tensile stress in the specimen and are independent of the mesh orientation. It is not known whether sets of As and Is that are all positive exist or not. However, the small fraction of beams that have negative values of A and/or I do not adversely affect either the computed elastic moduli or simulated fracture patterns.

A random geometry lattice can now represent an elastically homogeneous medium. The lattice can also be used to simulate heterogeneous media by using an overlay of a microstructure for implementing heterogeneity. The difficulty that arises then is how to chose the different properties in the various regions of the microstructure. An approach that combines using image techniques on real materials, and numerical procedures such as those explained in Ref. [14], seems to be very promising in this respect.

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